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The Kirchhoff indices of join networks

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ABSTRACT

In this work we compute *the effective resistance between any pair of vertices with respect to a value* $\lambda \geq 0$ *and a weight* ω on the vertex set of the *join network* in terms of the corresponding parameters for the factors. In particular, this allows us to express the Kirchhoff index with respect to a value $\lambda \geq 0$ and a weight ω of the join network in terms of the Kirchhoff indices of the factors. Moreover, we obtain a full description of the eigenvalues and the corresponding eigenfunctions for a positive semi-definite Schrödinger operator on a join network. Finally, we compute the effective resistances and the generalized Kirchhoff index with respect to a value $\lambda \geq 0$ and a weight ω for some families of join networks, specifically for star and cone networks.

1. Preliminaries

The Kirchhoff index was introduced in chemistry as a better alternative to other parameters used for discriminating among different molecules with similar shapes and structures; see [9]. This index is defined as the sum of all the effective resistances between any pair of vertices of the network and is also known as the Total Resistance; [8]. In view of its applications, a new line of research with a considerable amount of production has been developed and the Kirchhoff index has been computed for some classes of graphs with symmetries; see for instance [4,11] and the references therein. Moreover, it is of interest to calculate this parameter for composite networks and to find possible relations between the Kirchhoff indices of the original networks and those of their composite networks; see for instance [12].

In [5,6] the authors introduced a generalization of the Kirchhoff index of a finite network that consists in defining the effective resistance between any pair of vertices with respect to a value $\lambda \geq 0$ and a weight ω on the vertex set. It turns out that λ is the lowest eigenvalue of a suitable semi-definite positive Schrödinger operator and ω is the associated eigenfunction. Here we show that this generalization is essential to obtain the expression for the Kirchhoff index of a *join network* in terms of the Kirchhoff indices of the factors. Specifically, we obtain that the standard Kirchhoff index of a join network is the sum of the generalized Kirchhoff index of each factor plus a constant that depends only on the join conductances.

In this work the role of Green's function to evaluate the effective resistances of the network is essential. So, after the introduction of the main definitions of the involved operators and their properties, we get the expression of Green's function of a join network in terms of Green's functions of the factors. Therefore, as a by-product we obtain the expression of the effective resistances, and hence of the Kirchhoff index, of a join network in terms of the corresponding parameters of each factor network. Moreover, we also study the relation between the eigenvalues of the factor networks and those of their join network.

Given a finite set V, the set of real valued functions on V is denoted by $\mathcal{C}(V)$. The standard inner product on $\mathcal{C}(V)$ is denoted by $\langle \cdot, \cdot \rangle$ and hence if $u, v \in \mathcal{C}(V)$ then $\langle u, v \rangle = \sum_{x \in V} u(x)v(x)$. For any $x \in V$, $\varepsilon_x \in \mathcal{C}(V)$ stands for the Dirac

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function at x and 1 is the function defined by $1(x) = 1$, for any $x \in V$. On the other hand, $\omega \in \mathcal{C}(V)$ is called a *weight* if it satisfies that $\omega(x) > 0$ for any $x \in V$ and, moreover, $\langle \omega, \omega \rangle = 1$. The set of weights on V is denoted by $\Omega(V)$.

The triple $\Gamma = (V, E, c)$ denotes a finite network; that is, a finite connected graph without loops nor multiple edges, with vertex set V, whose cardinality equals n, and edge set E, in which each edge $\{x, y\}$ has been assigned a *conductance* $c(x, y) > 0$. So, the conductance can be considered as a symmetric function $c: V \times V \longrightarrow [0, +\infty)$ such that $c(x, x) = 0$ for any $x \in V$ and, moreover, vertex x is adjacent to vertex y iff $c(x, y) > 0$.

The *combinatorial Laplacian* or simply the *Laplacian* of the network Γ is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y)\Big(u(x) - u(y)\Big), \quad x \in V. \tag{1}$$

Given $q \in \mathcal{C}(V)$, the *Schrödinger operator* on Γ with *potential* q is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$, where $qu \in \mathcal{C}(V)$ is defined as $(qu)(x) = q(x)u(x)$; see for instance [2,7]. It is well-known that any Schrödinger operator is self-adjoint and we are interested in those Schrödinger operators that are positive semi-definite. The characterization of this type of operators was obtained in [2] by considering, for any $\omega \in \Omega(V)$, the *potential determined by* ω defined as the function $q_\omega = -\omega^{-1}\mathcal{L}(\omega)$.

Proposition 1.1 ([2, Prop. 3.3]). *The Schrödinger operator $\mathcal{L}_q$ is positive semi-definite iff there exist $\omega \in \Omega(V)$ and $\lambda \geq 0$ such that $q = q_\omega + \lambda$. Moreover, ω and λ are uniquely determined. In addition, $\mathcal{L}_q$ is singular iff $\lambda = 0$, in which case $\langle \mathcal{L}_{q_\omega}(v), v \rangle = 0$ iff $v = a\omega$, $a \in \mathbb{R}$. In any case, λ is the lowest eigenvalue of $\mathcal{L}_q$ and its associated eigenfunctions are multiple of ω.*

If $\mathcal{L}_q$ is positive definite, then it is invertible and its inverse is called *Green's operator*. On the other hand, when $\mathcal{L}_q$ is positive semi-definite and singular the operator that assigns to each function $f \in \mathcal{C}(V)$ the unique $u \in \mathcal{C}(V)$ such that $\mathcal{L}_q(u) = f - \langle \omega, f \rangle \omega$ and $\langle u, \omega \rangle = 0$ is called *Green's operator*. In any case, the Green operator is denoted by $\mathcal{G}_q$, see [5]. Moreover, the function $G_q : V \times V \longrightarrow \mathbb{R}$, defined as $G_q(x, y) = \mathcal{G}_q(\varepsilon_y)(x)$, for any $x, y \in V$, is called *Green's function*. Note that $\mathcal{G}_q(\omega) = \Lambda(\lambda)\omega$, where $\Lambda(\lambda) = \lambda^{-1}$ when $\lambda > 0$ and $\Lambda(0) = 0$.

In [5,6], the authors introduced a generalization of the concept of Kirchhoff index by defining the effective resistance and the Kirchhoff index with respect to a value $\lambda \geq 0$ and a weight $\omega \in \Omega(V)$. Specifically, we consider the functional on $\mathcal{C}(V)$ defined as

$$\mathfrak{I}_{x,y}(u) = 2\left[\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right] - \langle \mathcal{L}_q(u), u \rangle \tag{2}$$

and then we can give the following definition.

Definition 1.2. Given $x, y \in V$, the *effective resistance between x and y with respect to λ and ω* is the value

$$R_{\lambda,\omega}(x, y) = \max_{u \in \mathcal{C}(V)} \{\mathfrak{I}_{x,y}(u)\}.$$

Moreover, the *Kirchhoff index of Γ with respect to λ and ω* is the value

$$\mathsf{k}(\lambda, \omega) = \frac{1}{2}\sum_{x,y \in V} R_{\lambda,\omega}(x, y)\omega^2(x)\omega^2(y).$$

In addition, if we consider the functional

$$\mathfrak{I}_x(u) = 2\left[\frac{u(x)}{\omega(x)} - \langle u, \omega \rangle\right] - \langle \mathcal{L}_q(u), u \rangle \tag{3}$$

the *total resistance at $x \in V$ with respect to λ and ω* is defined as

$$r_{\lambda,\omega}(x) = \max_{u \in \mathcal{C}(V)} \{\mathfrak{I}_x(u)\}.$$

In the sequel we omit the expression with respect to λ and ω when it does not lead to confusion. When $\lambda = 0$ we usually omit the subindex λ in the above expressions and when ω is constant we also omit the subindex ω. Therefore, R is nothing else than the standard effective resistance of the network, whereas k is the Kirchhoff index introduced in the context of organic chemistry, see for instance [10].

The following formulas, that express the different parameters in terms of Green's functions, will be crucial in obtaining the main results of the present paper, see [5] for the proofs.

Proposition 1.3 ([5, Prop. 4.3]). *For any $x, y \in V$ it is verified that*

$$r_{\lambda,\omega}(x) = \frac{G_q(x,x)}{\omega^2(x)} - \Lambda(\lambda) \quad \text{and} \quad R_{\lambda,\omega}(x,y) = \frac{G_q(x,x)}{\omega^2(x)} + \frac{G_q(y,y)}{\omega^2(y)} - \frac{2G_q(x,y)}{\omega(x)\omega(y)}.$$

Therefore,

$$\mathsf{k}(\lambda,\omega) = \sum_{x\in V} r_{\lambda,\omega}(x)\omega^2(x) = \sum_{x\in V} G_q(x,x) - \Lambda(\lambda)$$

and, in particular,

$$\sum_{y\in V}\Big[R_{\lambda,\omega}(x,y) - \mathsf{k}(\lambda,\omega)\Big]\omega^2(y) = r_{\lambda,\omega}(x).$$

It is easy to conclude that when V is a singleton, then $\mathcal{G}_q(u) = \Lambda(\lambda)u$, for any $u \in \mathcal{C}(V)$, and, moreover, $r_{\lambda,\omega}$, $R_{\lambda,\omega}$ and $\mathsf{k}(\lambda,\omega)$ are null.

On the other hand, the above parameters can also be expressed in terms of the eigenvalues of $\mathcal{L}_q$ and their corresponding eigenfunctions. Specifically, if $\lambda = \lambda_0 < \lambda_1 \le \cdots \le \lambda_{n-1}$ are the eigenvalues of $\mathcal{L}_q$ and $\{u_j\}_{j=0}^{n-1}$, where $u_0 = \omega$, are the corresponding orthonormal basis of eigenfunctions, the following result holds.

Proposition 1.4 ([6, Prop. 3.4]). *For any $x, y \in V$ it is verified that*

$$r_{\lambda,\omega}(x) = \frac{1}{\omega^2(x)}\sum_{j=1}^{n-1}\frac{u_j^2(x)}{\lambda_j} \quad \text{and} \quad R_{\lambda,\omega}(x,y) = \sum_{j=1}^{n-1}\frac{1}{\lambda_j}\left(\frac{u_j(x)}{\omega(x)} - \frac{u_j(y)}{\omega(y)}\right)^2.$$

Therefore,

$$\mathsf{k}(\lambda,\omega) = \sum_{j=1}^{n-1}\frac{1}{\lambda_j}.$$

Observe that if $xQ_{q_\omega}(x)$ is the characteristic polynomial of $\mathcal{L}_{q_\omega}$, then $(x-\lambda)Q_{q_\omega}(x-\lambda)$ is the characteristic polynomial of $\mathcal{L}_q$. Therefore,

$$\mathsf{k}(\lambda,\omega) = -\frac{Q'_{q_\omega}(-\lambda)}{Q_{q_\omega}(-\lambda)}.$$

Moreover, if $Q_{q_\omega}(x) = a_n x^{n-1} + \cdots + a_2 x + a_1$, then $\mathsf{k}(\omega) = -\frac{a_2}{a_1}$.

2. Join networks

The join $\Gamma = \Gamma_1 + \Gamma_2$ of two graphs Γ_1 and Γ_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph Γ_1 union Γ_2 together with all the edges joining V_1 and V_2. For this composite graph the structure of the eigenvalues and eigenfunctions of the combinatorial Laplacian are well-known and the Kirchhoff index has been studied, see for instance [7,12]. In this section we consider the generalization of the join graph to the case of $(m+1)$-networks and we obtain the expression for Green's function, the eigenvalue and eigenfunctions, the effective resistances and the Kirchhoff index with respect to a non-negative value and a weight in terms of the corresponding parameter of the factors.

Let $\Gamma_i = (V_i, E_i, c_i)$, $i = 0, \dots, m$, be connected networks and $V = \coprod_{i=0}^m V_i$ the disjoint union of all vertex sets. We also consider $\omega \in \Omega(V)$ and $a_1, \dots, a_m > 0$.

We call *join network with base Γ_0, join conductances $\{a_i\}_{i=1}^m$ and weight ω*, the network $\Gamma = (V, E, c)$ obtained by joining the networks Γ_i, $i = 1, \dots, m$ to Γ_0; that is, the network whose conductance is given by $c(x,y) = c_i(x,y)$, for any $x, y \in V_i$, $i = 0, \dots, m$, by $c(x,y) = a_i\omega(x)\omega(y)$, for any $x \in V_0$ and $y \in V_i$, $i = 1, \dots, m$, and by $c(x,y) = 0$, otherwise. We can suppose, without loss of generality, that $a_1 \le \cdots \le a_m$.

Consider, for any $i = 0, \dots, m$, the value $\sigma_i = \left(\sum_{x\in V_i}\omega(x)^2\right)^{\frac{1}{2}}$. Then, given $i = 0, \dots, m$, if for any $x \in V_i$ we define $\omega_i(x) = \sigma_i^{-1}\omega(x)$, it is clear that $\omega_i \in \Omega(V_i)$. Moreover, for any $i = 0, \dots, m$, we identify $\mathcal{C}(V_i)$ with the subspace of $\mathcal{C}(V)$ formed by the functions that are null on $V \setminus V_i$. On the other hand, if $u \in \mathcal{C}(V)$, the restriction of u to V_i, $i = 0, \dots, m$ is also denoted by u. Observe that if $u \in \mathcal{C}(V_i)$ and $v \in \mathcal{C}(V)$, then $\langle u, v\rangle = \sum_{x\in V_i} u(x)v(x)$ and, in particular, $\langle u, v\rangle = 0$ when $v \in \mathcal{C}(V_j)$ with $j \ne i$.

If $\mathcal{L}$ is the combinatorial Laplacian of the join network Γ and, for $i = 0, \dots, m$, $\mathcal{L}^i$ denotes the combinatorial Laplacian of the network Γ_i, then for any $u \in \mathcal{C}(V)$ it holds that

$$\begin{aligned}
\mathcal{L}(u)(x) &= \mathcal{L}^0(u)(x) + \omega_0(x) \sum_{j=1}^m a_j\sigma_j\sigma_0\Big(u(x)\langle\omega_j, 1\rangle - \langle\omega_j, u\rangle\Big), \quad x \in V_0,\\
\mathcal{L}(u)(x) &= \mathcal{L}^i(u)(x) + \omega_i(x)a_i\sigma_i\sigma_0\Big(u(x)\langle\omega_0, 1\rangle - \langle\omega_0, u\rangle\Big), \quad x \in V_i,\ 1 \le i \le m.
\end{aligned} \tag{4}$$

Lemma 2.1. *Consider the potentials $q_\omega = -\omega^{-1}\mathcal{L}(\omega)$ on V and $q_{\omega_i} = -\omega_i^{-1}\mathcal{L}^i(\omega_i)$ on V_i, for $i = 0, \dots, m$. Then, $q_\omega = q_{\omega_0} + \sum_{j=1}^m a_j\sigma_j\Big(\sigma_j - \sigma_0\langle\omega_j, 1\rangle\omega_0\Big)$ on V_0 and, for any $i = 1, \dots, m$, $q_\omega = q_{\omega_i} + a_i\sigma_0\Big(\sigma_0 - \sigma_i\langle\omega_0, 1\rangle\omega_i\Big)$ on V_i.*

In the sequel we consider fixed $\lambda \ge 0$ and the potential on Γ given by $q = q_\omega + \lambda$. Moreover, we define the positive values $\gamma_0 = \lambda + \sum_{j=1}^m a_j\sigma_j^2$, $\gamma_i = \lambda + a_i\sigma_0^2$, $1 \le i \le m$, and the potentials on Γ_i given by $p_i = q_{\omega_i} + \gamma_i$, for any $0 \le i \le m$. Therefore, we get that

$$\begin{aligned}
\mathcal{L}_q(u) &= \mathcal{L}^0_{p_0}(u) - \left(\sum_{j=1}^m a_j\sigma_j\sigma_0\langle\omega_j, u\rangle\right)\omega_0, \quad \text{on } V_0,\\
\mathcal{L}_q(u) &= \mathcal{L}^i_{p_i}(u) - a_i\sigma_i\sigma_0\langle\omega_0, u\rangle\omega_i, \quad \text{on } V_i,\ i = 1, \dots, m.
\end{aligned} \tag{5}$$

2.1. Eigenvalues and eigenfunctions

The eigenvalues of the join of two graphs are known in terms of the eigenvalues of the factors, see [7, Prop. 4.11]. In this section we study the general case. First, we use Identities (5), to easily obtain most of the eigenvalues and eigenfunctions of $\mathcal{L}_q$ in terms of the eigenvalues and eigenfunctions of $\mathcal{L}^i_{p_i}$, $i = 0, \dots, m$.

Lemma 2.2. *Given $i = 0, \dots, m$ and $\gamma > \gamma_i$, if $u \in \mathcal{C}(V_i)$ satisfies $\mathcal{L}^i_{p_i}(u) = \gamma u$, then $\mathcal{L}_q(u) = \gamma u$.*

Proof. It suffices to observe that $\langle u, \omega_i\rangle = 0$, since $\gamma > \gamma_i$, and that $\langle u, \omega_j\rangle = 0$, for any $j = 0, \dots, m$ with $j \ne i$, since $u \in \mathcal{C}(V_i)$ and $\omega_j \in \mathcal{C}(V_j)$. $\square$

On the other hand, λ is the lowest eigenvalue of $\mathcal{L}_q$ whose corresponding eigenfunction is ω. Therefore, λ and any of the eigenvalues of $\mathcal{L}_q$ given in the above lemma are called *elemental eigenvalues*, whereas their corresponding eigenfunctions are called *elemental eigenfunctions*. So, the elemental eigenvalues of the join network Γ, other than λ, are the eigenvalues of each network Γ_i greater than its lowest eigenvalue γ_i. In addition, the elemental eigenfunctions, other than ω, are obtained by extending by zero the eigenfunctions of Γ_i, except the multiples of ω_i. To obtain the non-elemental eigenvalues we need to introduce the polynomial

$$P_\Gamma(x) = \prod_{j=1}^m(\gamma_j - x) + \sum_{i=1}^m a_i\sigma_i^2 \prod_{\substack{j=1\\ j\ne i}}^m(\gamma_j - x)$$

that is called the *join polynomial of the network Γ*.

Lemma 2.3. *All roots of P_Γ are real. Moreover, if $\lambda_1 \le \cdots \le \lambda_m$ are the roots of P_Γ, counting with their multiplicity, the following properties hold:*

1. *$\gamma_1 \le \lambda_1 \le \gamma_2 \le \cdots \le \lambda_{m-1} \le \gamma_m \le \lambda_m \le \lambda + a_m$.*
2. *$\lambda_m = \lambda + a_m$ iff $a_1 = \cdots = a_m$, in which case the roots of P_Γ are $\lambda + a_m\sigma_0^2$ and $\lambda + a_m$.*
3. *If for any $i = 1, \dots, m$ we consider $I(i) = \{j = 1, \dots, m : a_j = a_i\}$ and $m_i = |I(i)|$, then γ_i is a root of order $m_i - 1$ of P_Γ.*
4. *If $\lambda_j \ne \gamma_i$, for any $i = 1, \dots, m$, then λ_j is a simple root of P_Γ.*

Proof. Simple algebraic computations show that the characteristic polynomial of the symmetric matrix

$$M_\Gamma = \begin{bmatrix} \gamma_0 & -a_1\sigma_0\sigma_1 & \cdots & -a_m\sigma_0\sigma_m \\ -a_1\sigma_0\sigma_1 & \gamma_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_m\sigma_0\sigma_m & 0 & \cdots & \gamma_m \end{bmatrix},$$

is $Q(x) = \prod_{j=0}^{m}(\gamma_j - x) - \sigma_0^2 \sum_{i=1}^{m} a_i^2\sigma_i^2 \prod_{\substack{j=1\\ j\neq i}}^{m}(\gamma_j - x)$. Then, taking into account that $a_i\sigma_0^2 = (\gamma_i - x) + (x - \lambda)$, we have that

$$Q(x) = \prod_{j=1}^{m}(\gamma_j - x)\left[\gamma_0 - x - \sum_{i=1}^{m} a_i\sigma_i^2\right] + (\lambda - x)\sum_{i=1}^{m} a_i\sigma_i^2 \prod_{\substack{j=1\\ j\neq i}}^{m}(\gamma_j - x) = (\lambda - x)P_\Gamma(x).$$

Therefore, $P_\Gamma(x)$ has m real roots. Moreover, $P_\Gamma(x) > 0$ when $x < \gamma_1$ and hence,

$$\lambda < \gamma_1 \le \lambda_1 \le \gamma_2 \le \cdots \le \lambda_{m-1} \le \gamma_m \le \lambda_m,$$

follows from the Cauchy Interlace Theorem, taking into account that $\lambda < \gamma_1$.

If $x \ge \lambda + a_m$, then for any $i = 1, \ldots, m$ we get that $x > \gamma_i$ and

$$x - \gamma_i \ge \lambda + a_m - \gamma_i = a_m - a_i\sigma_0^2 \ge a_i(1 - \sigma_0^2)$$

with equality iff $x = \lambda + a_m$ and $a_i = a_m$. So, we get that

$$(-1)^m P_\Gamma(x) = \prod_{j=1}^{m}(x - \gamma_j)\left[1 - \sum_{i=1}^{m}\frac{a_i\sigma_i^2}{(x - \gamma_i)}\right] \ge \prod_{j=1}^{m}(x - \gamma_j)\left[1 - \sum_{i=1}^{m}\frac{\sigma_i^2}{(1 - \sigma_0^2)}\right] = 0$$

with equality iff $x = \lambda + a_m$ and $a_1 = \cdots = a_m$. In conclusion, $\lambda_m \le \lambda + a_m$, with equality iff $a_1 = \cdots = a_m$. Moreover this last condition is equivalent to $\gamma_1 = \cdots = \gamma_m$ and, then, the Cauchy Interlace Theorem implies that $\lambda_1 = \cdots = \lambda_{m-1} = \gamma_1 = \gamma_m < \lambda + a_m$.

On the other hand, for any $i = 1, \ldots, m$, we get that γ_i is root of order m_i of $\prod_{j=1}^{m}(\gamma_j - x)$ and hence it is a root of order $m_i - 1$ of P_Γ, since it is a root of order $m_i - 1$ of the polynomial $\sum_{i=1}^{m} a_i\sigma_i^2 \prod_{\substack{j=1\\ j\neq i}}^{m}(x - \gamma_j)$. Finally, Claim 4 is a straightforward consequence of the Cauchy Interlace Theorem. □

Proposition 2.4. *The non-elemental eigenvalues of $\mathcal{L}_q$ are the roots of the join polynomial of Γ. Moreover, if $P_\Gamma(\gamma) = 0$, then the following properties hold:*

1. *If $\gamma \neq \gamma_i,\ i = 1, \ldots, m$, then γ is simple and a corresponding unitary eigenfunction is given by*

$$u = \left(1 + \sigma_0^2 \sum_{j=1}^{m}\frac{a_j^2\sigma_j^2}{(\gamma_j - \gamma)^2}\right)^{-\frac{1}{2}}\left(\omega_0 + \sigma_0\sum_{j=1}^{m}\frac{a_j\sigma_j\omega_j}{\gamma_j - \gamma}\right).$$

2. *If $\gamma = \gamma_i$ for some $i = 1, \ldots, m$ and $\gamma \neq \gamma_{i-1}$, then an orthonormal basis of the subspace of eigenfunctions corresponding to γ_i is given by*

$$u_k = \left(\sum_{j=0}^{k-1}\sigma_{i+j}^2\right)^{-\frac{1}{2}}\left(\sum_{j=0}^{k}\sigma_{i+j}^2\right)^{-\frac{1}{2}}\sum_{j=0}^{k-1}\sigma_{i+j}\Big(\sigma_{i+j}\omega_{i+k} - \sigma_{i+k}\omega_{i+j}\Big), \quad k = 1, \ldots, m_i - 1.$$

Proof. If we consider $u = \sum_{j=0}^{m}\tau_j\omega_j$, then $\tau_j = \langle\omega_j, u\rangle$, for any $j = 0, \ldots, m$. Therefore, from Equalities (5) we get that $\mathcal{L}_q(u) = \gamma u$ iff

$$\gamma_0\tau_0\omega_0 - \left(\sum_{j=1}^{m} a_j\sigma_j\sigma_0\tau_j\right)\omega_0 = \gamma\tau_0\omega_0, \quad \text{on } V_0,$$
$$\gamma_i\tau_i\omega_i - a_i\sigma_i\sigma_0\tau_0\omega_i = \gamma\tau_i\omega_i, \quad \text{on } V_i,\ i = 1, \ldots, m$$

or, in an equivalent manner, iff

$$(\gamma_0 - \gamma)\tau_0 = \sum_{j=1}^{m} a_j\sigma_j\sigma_0\tau_j \quad\text{and}\quad (\gamma_i - \gamma)\tau_i = a_i\sigma_i\sigma_0\tau_0, \quad i = 1, \ldots, m. \tag{E1}$$

The above equalities are equivalent to γ being an eigenvalue of the matrix M_Γ defined in the proof of Lemma 2.3. Therefore, if $\gamma \neq \lambda$, then γ is a root of the join polynomial of Γ. Moreover, the values $\tau_0, \ldots, \tau_m$ must be the components of an eigenvector of M_Γ corresponding to γ. In addition, if $(\tau_0, \ldots, \tau_m)^T$ and $(\hat\tau_0, \ldots, \hat\tau_m)^T$ are linearly independent, respectively orthogonal, eigenvectors of M_Γ, then $u = \sum_{j=0}^{m}\tau_j\omega_j$ and $\hat u = \sum_{j=0}^{m}\hat\tau_j\omega_j$ are linearly, respectively orthogonal, eigenfunctions of $\mathcal{L}_q$. Therefore, all the roots of P_Γ, counting with their multiplicities, are non-elemental eigenvalues of $\mathcal{L}_q$.

(1) If $P_\Gamma(\gamma) = 0$ and $\gamma \neq \gamma_i$ for any $i = 1, \dots, m$, then Lemma 2.3 implies that γ is a simple eigenvalue of $\mathcal{L}_q$. Moreover the last equations in (E1) imply that $\tau_i = \frac{a_i\sigma_i\sigma_0\tau_0}{\gamma_i-\gamma}$, $i = 1, \dots, m$. In addition, as $\langle u, u\rangle = \sum_{j=0}^m \tau_j^2$, then u is unitary iff

$$\tau_0^2 = \left(1 + \sigma_0^2 \sum_{j=1}^m \frac{a_i^2\sigma_i^2}{(\gamma_i - \gamma)^2}\right)^{-1}.$$

(2) When $\gamma = \gamma_i$ for some $i = 1, \dots, m$, then the i-th equation in (E1) implies that $\tau_0 = 0$, whereas the j-th equation for $j \notin I(i)$ implies that $\tau_j = 0$. In addition, the first equation in (E1) implies that u is an eigenfunction corresponding to γ iff $\sum_{j\in I(i)} \sigma_j\tau_j = 0$, since $a_j = a_i$ for any $j \in I(i)$.

Moreover, $I(i) = \{i, \dots, i + m_i\}$, since $\gamma \neq \gamma_{i-1}$, and hence

$$v_k = \sigma_i\omega_{i+k} - \sigma_{i+k}\omega_i, \quad k = 1, \dots, m_i - 1$$

is a basis of the subspace of eigenfunctions corresponding to γ_i. The claimed orthonormal basis is obtained by applying the Gram–Schmidt process to the above system. □

In the following result we specify the standard case; that is, the case of the join of two graphs, $\Gamma_0 = (V_0, E_0)$ and $\Gamma_1 = (V_1, E_1)$ with $|V_0| = n_0$ and $|V_1| = n_1$. This corresponds to taking $m = 1, \omega = \frac{1}{\sqrt{n_0+n_1}}$ and $a_1 = n_0 + n_1$, so that $c(x, y) = 1$ for any $x \in V_0$ and $y \in V_1$. Let $0 = \nu_1 < \nu_2 \le \cdots \le \nu_{n_0}$ and $0 = \mu_1 < \mu_2 \le \cdots \le \mu_{n_1}$ be the eigenvalues of the combinatorial Laplacians of Γ_0 and Γ_1, respectively. In addition, let $\left\{n_0^{-\frac{1}{2}}1, u_2, \dots, u_{n_0}\right\} \subset \mathcal{C}(V_0)$ and $\left\{n_1^{-\frac{1}{2}}1, v_2, \dots, v_{n_1}\right\} \subset \mathcal{C}(V_1)$ be the corresponding orthonormal basis of eigenfunctions.

Corollary 2.5. *The eigenvalues of the combinatorial Laplacian $\mathcal{L}$ of the join graph are*

$$0, \nu_2 + n_1, \dots, \nu_{n_0} + n_1, \mu_2 + n_0, \dots, \mu_{n_1} + n_0, n_0 + n_1$$

and the corresponding orthonormal basis of eigenfunctions is

$$(n_0 + n_1)^{-\frac{1}{2}}1, u_2, \dots, u_{n_0}, v_2, \dots, v_{n_1}, u,$$

where $u(x) = \frac{n_1}{\sqrt{n_0n_1(n_0+n_1)}}$ if $x \in V_0$ and $u(x) = -\frac{n_0}{\sqrt{n_0n_1(n_0+n_1)}}$ if $x \in V_1$.

The above result is well-known and it can be found in [7] and references therein.

2.2. Green's function and effective resistances

The main objective in this section is to obtain Green's function of the join network in terms of Green's functions of the factors. As a by-product we also obtain the effective resistances and the Kirchhoff index, with respect to a non-negative value and a weight, in terms of the corresponding parameters of each factor network.

Throughout the section $\mathcal{G}_p^i$ will denote Green's operator for $\mathcal{L}_p^i$ on Γ_i. We define the value $\alpha = \sum_{j=1}^m \gamma_j^{-1}\sigma_j^2$, since it appears repeatedly throughout the paper.

Lemma 2.6. *Given $f \in \mathcal{C}(V)$, then*

$$\sum_{j=1}^m \gamma_j^{-1}a_j\sigma_j\langle\omega_j, f\rangle = \frac{1}{\sigma_0^2}\left[\langle\omega, f\rangle - \sigma_0\langle\omega_0, f\rangle - \lambda\sum_{j=1}^m \gamma_j^{-1}\sigma_j\langle\omega_j, f\rangle\right].$$

In particular, $\sum_{j=1}^m \gamma_j^{-1}a_j\sigma_j^2 = \frac{1-\lambda\alpha}{\sigma_0^2} - 1$ and $\sum_{j=1}^m \gamma_j^{-1}a_j^2\sigma_j^2 = \frac{\gamma_0}{\sigma_0^2} - \frac{\lambda(1-\lambda\alpha)}{\sigma_0^4}$.

Proof. The first part is a consequence of the identity $a_j = \frac{\gamma_j-\lambda}{\sigma_0^2}$ for any $j = 1, \dots, m$ and the last equality follows by re-writing the sum as

$$\sum_{j=1}^m \gamma_j^{-1}a_j^2\sigma_j^2 = \frac{1}{\sigma_0^2}\sum_{j=1}^m (1 - \lambda\gamma_j^{-1})a_j\sigma_j^2. \quad \square$$

For any $f \in \mathcal{C}(V)$ consider the value

$$P(f) = \left[\Lambda(\lambda) + \frac{\alpha}{1 - \lambda\alpha}\right]\langle\omega, f\rangle - \frac{1}{1 - \lambda\alpha}\sum_{j=1}^m \gamma_j^{-1}\sigma_j\langle\omega_j, f\rangle$$

and also the following projectors

$$\mathcal{P}_0(f) = \Big(\sigma_0 P(f) - \gamma_0^{-1}\langle \omega_0, f\rangle\Big)\omega_0 \quad \text{and} \quad \mathcal{P}_i(f) = \sigma_0^2\gamma_i^{-1}a_i\sigma_i P(f)\,\omega_i, \quad i = 1, \dots, m.$$

Proposition 2.7. *Let $f \in \mathcal{C}(V)$ such that $\langle \omega, f\rangle = 0$ when $\lambda = 0$ and consider the Poisson equation on V, $\mathcal{L}_q(u) = f$. Then, the function*

$$u = \mathcal{G}^i_{p_i}(f) + \mathcal{P}_i(f) \quad \text{on } V_i,\ i = 0, \dots, m$$

is the unique solution of the Poisson equation when $\lambda > 0$ or the unique solution of the Poisson equation such that $\langle \omega, u\rangle = 0$ when $\lambda = 0$.

Proof. From Identities (5) we get that $\mathcal{L}_q(u) = f$ on V iff

$$\mathcal{L}^0_{p_0}(u) = f + \left(\sum_{j=1}^m a_j\sigma_j\sigma_0\langle \omega_j, u\rangle\right)\omega_0, \quad \text{on } V_0,$$
$$\mathcal{L}^i_{p_i}(u) = f + a_i\sigma_i\sigma_0\langle \omega_0, u\rangle\omega_i, \quad \text{on } V_i,\ i = 1, \dots, m.$$

Therefore,

$$\begin{aligned}\gamma_0\langle \omega_0, u\rangle &= \langle \omega_0, \mathcal{L}^0_{p_0}(u)\rangle = \langle \omega_0, f\rangle + \sigma_0\sum_{j=1}^m a_j\sigma_j\langle \omega_j, u\rangle,\\ \gamma_i\langle \omega_i, u\rangle &= \langle \omega_i, \mathcal{L}^i_{p_i}(u)\rangle = \langle \omega_i, f\rangle + a_i\sigma_i\sigma_0\langle \omega_0, u\rangle, \quad i = 1, \dots, m.\end{aligned} \tag{S_0}$$

Multiplying by $a_i\sigma_i$ the i-th equation and summing we obtain the linear system

$$\begin{aligned}&\gamma_0\langle \omega_0, u\rangle - \sigma_0\sum_{j=1}^m a_j\sigma_j\langle \omega_j, u\rangle = \langle \omega_0, f\rangle,\\ &-\sigma_0\langle \omega_0, u\rangle\sum_{j=1}^m \gamma_j^{-1}a_j^2\sigma_j^2 + \sum_{j=1}^m a_j\sigma_j\langle \omega_j, u\rangle = \sum_{j=1}^m \gamma_j^{-1}a_j\sigma_j\langle \omega_j, f\rangle\end{aligned} \tag{S}$$

that from Lemma 2.6 implies

$$\gamma_0\sigma_0\langle \omega_0, u\rangle - \sigma_0^2\sum_{j=1}^m a_j\sigma_j\langle \omega_j, u\rangle = \sigma_0\langle \omega_0, f\rangle,$$
$$-\sigma_0\langle \omega_0, u\rangle\left[\gamma_0 - \frac{\lambda(1-\lambda\alpha)}{\sigma_0^2}\right] + \sigma_0^2\sum_{j=1}^m a_j\sigma_j\langle \omega_j, u\rangle = \langle \omega, f\rangle - \sigma_0\langle \omega_0, f\rangle - \lambda\sum_{j=1}^m \gamma_j^{-1}\sigma_j\langle \omega_j, f\rangle$$

and hence

$$\langle \omega_0, u\rangle\left[\frac{\lambda(1-\lambda\alpha)}{\sigma_0}\right] = \langle \omega, f\rangle - \lambda\sum_{j=1}^m \gamma_j^{-1}\sigma_j\langle \omega_j, f\rangle. \tag{E2}$$

When $\lambda > 0$, Eq. (E2) implies that the solution of the system (S) is given by

$$\langle \omega_0, u\rangle = \sigma_0 P(f) \quad \text{and} \quad \sum_{j=1}^m a_j\sigma_j\langle \omega_j, u\rangle = \gamma_0 P(f) - \frac{\langle \omega_0, f\rangle}{\sigma_0}.$$

Therefore, $\mathcal{L}_q(u) = f$ on V iff $\mathcal{L}^i_{p_i}(u) = f + \gamma_i\mathcal{P}_i(f)$ on V_i for any $i = 0, \dots, m$; that is, iff $u = \mathcal{G}^i_{p_i}(f) + \mathcal{P}_i(f)$ on V_i for any $i = 0, \dots, m$.

When $\lambda = 0$, Eq. (E2) implies that the Poisson equation has solution iff $\langle \omega, f\rangle = 0$. Then, keeping in mind that $\gamma_i = a_i\sigma_0^2$, from the last m equations of (S_0), we obtain that the unique solution such that $\langle \omega, u\rangle = 0$ verifies that

$$\langle \omega_0, u\rangle = -\sigma_0\sum_{j=1}^m \gamma_j^{-1}\sigma_j\langle \omega_j, f\rangle = \sigma_0 P(f) \quad \text{and} \quad \sum_{j=1}^m a_j\sigma_j\langle \omega_j, u\rangle = \gamma_0 P(f) - \frac{\langle \omega_0, f\rangle}{\sigma_0};$$

that is, $\mathcal{L}^i_{p_i}(u) = f + \gamma_i\mathcal{P}_i(f)$ on V_i for any $i = 0, \dots, m$, and the conclusion follows. $\square$

Theorem 2.8. *Green's function of the join network Γ is given by*

$$G_q = G^i_{p_i} + g_{ii}(\lambda)\omega\otimes\omega, \quad \text{on } V_i\times V_i,\ i=0,\dots,m,$$
$$G_q = g_{ij}(\lambda)\omega\otimes\omega, \quad \text{on } V_i\times V_j,\ i,j=0,\dots,m,\ j\neq i,$$

where, for any $i,j=1,\dots,m$,

$$\mathrm{g}_{00}(\lambda)=\Lambda(\lambda)+\frac{\alpha}{1-\lambda\alpha}-\frac{1}{\sigma_0^2\gamma_0}, \qquad \mathrm{g}_{0i}(\lambda)=\mathrm{g}_{i0}(\lambda)=\Lambda(\lambda)+\frac{\alpha}{1-\lambda\alpha}-\frac{1}{\gamma_i(1-\lambda\alpha)},$$

and

$$\mathrm{g}_{ij}(\lambda)=\Lambda(\lambda)+\frac{\alpha}{1-\lambda\alpha}+\frac{\lambda-\gamma_i-\gamma_j}{\gamma_i\gamma_j(1-\lambda\alpha)}.$$

Proof. When $\lambda>0$, $u=G_q(\cdot,y)$ is the unique solution of $\mathcal{L}_q(u)=\varepsilon_y$ on V. From Proposition 2.7, we get that $u=\mathcal{G}^i_{p_i}(\varepsilon_y)+\mathcal{P}_i(\varepsilon_y)$ on V_i, for any $i=0,\dots,m$. Therefore, if $y\in V_i$, then $u(x)=G^i_{p_i}(x,y)+\mathcal{P}_i(\varepsilon_y)(x)$ when $x\in V_i$ and $u(x)=\mathcal{P}_j(\varepsilon_y)(x)$ when $x\in V_j, j\neq i$.

The expression for Green's function of the join network follows by taking into account that for any $j=0,\dots,m$ and any $x\in V_j$, $\mathcal{P}_j(\varepsilon_y)(x)=\mathrm{g}_{ji}(\lambda)\omega(x)\omega(y)$.

When $\lambda=0$, $u=G_q(\cdot,y)$ is the unique solution of $\mathcal{L}_q(u)=\varepsilon_y-\omega(y)\omega$ on V such that $\langle\omega,u\rangle=0$. From Proposition 2.7, $u=\mathcal{G}^i_{p_i}(\varepsilon_y)+\mathcal{P}_i(\varepsilon_y)-\omega(y)\mathcal{P}_i(\omega)$ on V_i, for any $i=0,\dots,m$. Therefore, if $y\in V_i$, then $u(x)=G^i_{p_i}(x,y)+\mathcal{P}_i(\varepsilon_y)(x)-\omega(y)\mathcal{P}_i(\omega)(x)$ when $x\in V_i$ and $u(x)=\mathcal{P}_j(\varepsilon_y)(x)-\omega(y)\mathcal{P}_j(\omega)(x)$ when $x\in V_j,\ j\neq i$.

In conclusion, if $y\in V_i$, then

$$u=\mathcal{G}^0_{p_0}(\varepsilon_y)-\mathcal{P}(\varepsilon_y)-\frac{\langle\omega_0,\varepsilon_y\rangle}{\sigma_0\gamma_0}\omega+\alpha\omega(y)\omega \quad \text{on } V_0$$
$$u=\mathcal{G}^j_{p_j}(\varepsilon_y)-\mathcal{P}(\varepsilon_y)-\frac{1}{\gamma_j}\omega(y)\omega+\alpha\omega(y)\omega \quad \text{on } V_j,\ j=1,\dots,m$$

and the expression for Green's function follows, taking into account that $\mathcal{P}(\varepsilon_y)=0$ if $y\in V_0$ and that $\mathcal{P}(\varepsilon_y)=\gamma_i^{-1}\omega(y)\omega$ if $y\in V_i,\ i\neq 0$. $\square$

The following result gives the expression of the Kirchhoff index, the effective resistances and the total resistances with respect to a weight and a non-negative value in terms of the corresponding parameters for the networks that form the join network. These expressions follow directly from the formulas for the Kirchhoff index and for the effective resistances given in Proposition 1.3. In all the expressions, the superscript i in a parameter stands for the corresponding parameter on the network Γ_i.

Proposition 2.9. *If $\beta=\sum_{j=1}^m\gamma_j^{-2}\sigma_j^2$, then it is verified that*

$$\mathsf{k}(\lambda,\omega)=\sum_{j=0}^m\mathsf{k}_j(\gamma_j,\omega_j)+\sum_{j=1}^m\frac{1}{\gamma_j}+\frac{(\lambda\beta-\alpha)}{1-\lambda\alpha}.$$

In particular,

$$\sum_{j=1}^m\frac{1}{\lambda_j}=\sum_{j=1}^m\frac{1}{\gamma_j}+\frac{(\lambda\beta-\alpha)}{1-\lambda\alpha}=-\frac{P'_\Gamma(0)}{P_\Gamma(0)}.$$

Moreover, if $x\in V_i$, for any $i=0,\dots,m$, then

$$r_{\lambda,\omega}(x)=\sigma_i^{-2}r^i_{\gamma_i,\omega_i}(x)+\frac{\alpha}{1-\lambda\alpha}+t_i(\lambda),$$

where $t_0(\lambda)=0$ and $t_i(\lambda)=\frac{1}{\gamma_i\sigma_i^2}+\frac{\lambda-2\gamma_i}{\gamma_i^2(1-\lambda\alpha)}$, for any $i=1,\dots,m$. In addition, if $x,y\in V_i$, for any $i=0,\dots,m$, then

$$R_{\lambda,\omega}(x,y)=\sigma_i^{-2}R^i_{\gamma_i,\omega_i}(x,y)$$

whereas, if $x\in V_i, y\in V_j$ for any $i,j=0,\dots,m$ with $i\neq j$, we get

$$R_{\lambda,\omega}(x,y)=\sigma_i^{-2}r^i_{\gamma_i,\omega_i}(x)+\sigma_j^{-2}r^j_{\gamma_j,\omega_j}(y)+\frac{1}{\gamma_j\sigma_j^2}+\frac{1}{\gamma_i\sigma_i^2}+s_{ij}(\lambda)$$

where

$$s_{0j}(\lambda)=\frac{\lambda}{\gamma_j^2(1-\lambda\alpha)}-\frac{1}{\gamma_0\sigma_0^2} \quad \text{and} \quad s_{ij}(\lambda)=\frac{\lambda(\gamma_i-\gamma_j)^2}{\gamma_i^2\gamma_j^2(1-\lambda\alpha)}, \quad i,j=1,\dots,m,\ i\neq j.$$

3. Star network

In this section we assume that the base network Γ_0 is a singleton; that is, $V_0 = \{x_0\}$, and for all $1 \le i \le m$ the network Γ_i is a copy of some network $\widehat{\Gamma}$. For any $1 \le i \le m$, we denote $V_i = \{x_{i1}, \dots, x_{in}\}$. Moreover, we consider $a > 0$, $0 < \sigma_0 < 1$, and $\omega \in \Omega(V)$ defined as $\omega(x_0) = \sigma_0$ and $\omega(x_{ij}) = \sqrt{\frac{1-\sigma_0^2}{nm}}$ for any $1 \le i \le m$ and $1 \le j \le n$. Then, the corresponding join network Γ is called *m-star network with join conductance a*.

With the notations of the preceding sections, $\gamma_i = \gamma = \lambda + a\sigma_0^2$ and $p_i = \gamma$, for any $i = 1, \dots, m$. Moreover, we denote by $\widehat{G}, \widehat{r}, \widehat{R}$ and $\widehat{\mathsf{k}}$, the Green's function, the total resistance, the effective resistance and the Kirchhoff index of the network $\widehat{\Gamma}$.

Proposition 3.1. *Green's function of a m-star network is given by*

$$G_q(x_0, x_0) = \frac{1-\sigma_0^2}{\lambda + a} + \Lambda(\lambda)\sigma_0^2$$

and for any $k, s = 1, \dots, n,\ i, j = 1, \dots, m,\ i \neq j$,

$$G_q(x_{ik}, x_{is}) = \widehat{G}_\gamma(x_{ik}, x_{is}) + \frac{(1-\sigma_0^2)}{nm}\left[\Lambda(\lambda) - \frac{(\lambda + a + a\sigma_0^2)}{(\lambda + a)(\lambda + a\sigma_0^2)}\right],$$

$$G_q(x_0, x_{ik}) = \frac{\sigma_0\sqrt{1-\sigma_0^2}}{\sqrt{mn}}\left[\Lambda(\lambda) - \frac{1}{\lambda + a}\right],$$

$$G_q(x_{ik}, x_{js}) = \frac{(1-\sigma_0^2)}{nm}\left[\Lambda(\lambda) - \frac{(\lambda + a + a\sigma_0^2)}{(\lambda + a)(\lambda + a\sigma_0^2)}\right].$$

Therefore,

$$\mathsf{k}(\lambda, \omega) = m\widehat{\mathsf{k}}(\gamma) + \frac{m-1}{\lambda + a\sigma_0^2} + \frac{1}{\lambda + a}.$$

Moreover, for any $i = 0, \dots, m$,

$$r_{\lambda,\omega}(x_0) = \frac{(1-\sigma_0^2)}{\sigma_0^2(\lambda + a)} \quad \text{and} \quad r_{\lambda,\omega}(x_{ik}) = \frac{1}{1-\sigma_0^2}\left[m\widehat{r}_\gamma(x_{ik}) + \frac{m-1}{\lambda + a\sigma_0^2} + \frac{\sigma_0^2}{\lambda + a}\right].$$

In addition, for any $k, s = 1, \dots, n, i, j = 1, \dots, m, i \neq j$,

$$R_{\lambda,\omega}(x_{ik}, x_{is}) = \frac{m}{1-\sigma_0^2}\widehat{R}_\gamma(x_{ik}, x_{is})$$

$$R_{\lambda,\omega}(x_0, x_{js}) = \frac{1}{1-\sigma_0^2}\left[m\widehat{r}_\gamma(x_{js}) + \frac{m-1}{\lambda + a\sigma_0^2} + \frac{1}{\sigma_0^2(\lambda + a)}\right]$$

$$R_{\lambda,\omega}(x_{ik}, x_{js}) = \frac{m}{1-\sigma_0^2}\left[\widehat{r}_\gamma(x_{ik}) + \widehat{r}_\gamma(x_{js}) + \frac{2}{\lambda + a\sigma_0^2}\right].$$

To end this section let us consider two particular cases of star networks; namely, the *m-star cycle* and the *m-star path* that are obtained when $\widehat{\Gamma}$ is a cycle and a path, respectively. We need to remember some properties of the *First and Second order Chebyshev Polynomials*, that are respectively defined by the following recurrences:

$$\begin{aligned} &T_0(x) = 1, \quad T_1(x) = x, \quad T_{k+2}(x) = 2xT_{k+1}(x) - T_k(x), \quad k \ge 0, \\ &U_{-2}(x) = -1, \quad U_{-1}(x) = 0, \quad U_k(x) = 2xU_{k-1}(x) - U_{k-2}(x), \quad k \ge 0. \end{aligned} \tag{6}$$

Moreover, for any $k \ge 0$ and any $x \in \mathbb{R}$, it is satisfied that $T_k(x) = xU_{k-1}(x) - U_{k-2}(x)$, $2(x-1)\sum_{l=0}^{k} U_l(x) = U_{k+1}(x) - U_k(x) - 1$, $T_k'(x) = kU_{k-1}(x)$ and also $(x^2-1)U_k'(x) = kT_{k+1}(x) - U_{k-1}(x)$. Thus,

$$\frac{kU_{k-1}(x)}{T_k(x) - 1} = \sum_{l=0}^{k-1} \frac{1}{x - \cos\left(\frac{2l\pi}{k}\right)} \quad \text{and} \quad \frac{kT_k(x) - xU_{k-1}(x)}{(x+1)U_{k-1}(x)} = \sum_{l=1}^{k-1} \frac{x-1}{x - \cos\left(\frac{l\pi}{k}\right)}$$

since $\left\{\cos\left(\frac{2l\pi}{k}\right)\right\}_{l=0}^{k-1}$ are the roots of the polynomial $T_k(x) - 1$, whereas $\left\{\cos\left(\frac{l\pi}{k}\right)\right\}_{l=1}^{k}$ are the roots of the polynomial $(x+1)U_{k-1}(x)$.

In the following lemma we show the expression for Green's function of a n-cycle with constant weight and constant conductance that can be easily deduced from Green's function of a path with periodic boundary conditions; see [3, Proposition 3.12].

Lemma 3.2. *If $c, \gamma > 0$, and $p = 1 + \frac{\gamma}{2c}$, then Green's function for the n-cycle with constant conductance c is given by*

$$\widetilde{G}_\gamma(x_k, x_s) = \frac{U_{n-1-|k-s|}(p) + U_{|k-s|-1}(p)}{2c(T_n(p) - 1)}.$$

Moreover,

$$\widetilde{R}_\gamma(x_k, x_s) = \frac{n\Big(U_{n-1}(p) - U_{n-1-|k-s|}(p) - U_{|k-s|-1}(p)\Big)}{c(T_n(p) - 1)}$$

and

$$\widetilde{\mathsf{k}}(\gamma) = \widetilde{r}_\gamma(x_k) = \frac{nU_{n-1}(p)}{2c(T_n(p) - 1)} - \frac{1}{\gamma} = \sum_{k=1}^{n-1} \frac{1}{\gamma + 4c\sin^2\left(\frac{k\pi}{n}\right)}.$$

On the other hand, Green's function for a n-path with constant conductance is given by the following result; see [3, Proposition 5.2].

Lemma 3.3. *If $c, \gamma > 0$, and $p = 1 + \frac{\gamma}{2c}$, then Green's function for the n-path with constant conductance c is given by*

$$\overline{G}_\gamma(x_k, x_s) = \frac{T_{n-|k-s|}(p) + T_{n-s-k+1}(p)}{\gamma(p+1)U_{n-1}(p)}.$$

Therefore,

$$\overline{R}_\gamma(x_k, x_s) = \frac{2n\Big(T_n(p) + T_{n+1-k-s}(p)(T_{|k-s|}(p) - 1) - T_{n-|k-s|}(p)\Big)}{\gamma(p+1)U_{n-1}(p)},$$

$$\overline{r}_\gamma(x_k) = \frac{n\Big(T_n(p) + T_{n+1-2k}(p)\Big)}{\gamma(p+1)U_{n-1}(p)} - \frac{1}{\gamma}$$

and

$$\overline{\mathsf{k}}(\gamma) = \frac{nT_n(p) - pU_{n-1}(p)}{\gamma(p+1)U_{n-1}(p)} = \sum_{k=1}^{n-1} \frac{1}{\gamma + 4c\sin^2\left(\frac{k\pi}{2n}\right)}.$$

From Proposition 3.1 we get the following expressions for a m-star cycle.

Proposition 3.4. *If $p = 1 + \frac{\lambda + a\sigma_0^2}{2c}$, then Green's function of a m-star cycle is given by*

$$G_q(x_0, x_0) = \frac{1 - \sigma_0^2}{\lambda + a} + \Lambda(\lambda)\sigma_0^2$$

and for any $k, s = 1, \ldots, n$, $i, j = 1, \ldots, m$, $i \neq j$,

$$G_q(x_{ik}, x_{is}) = \frac{U_{n-1-|k-s|}(p) + U_{|k-s|-1}(p)}{2c(T_n(p) - 1)} + \frac{(1 - \sigma_0^2)}{nm}\left[\Lambda(\lambda) - \frac{(\lambda + a + a\sigma_0^2)}{(\lambda + a)(\lambda + a\sigma_0^2)}\right],$$

$$G_q(x_0, x_{ik}) = \frac{\sigma_0\sqrt{1 - \sigma_0^2}}{\sqrt{mn}}\left[\Lambda(\lambda) - \frac{1}{\lambda + a}\right],$$

$$G_q(x_{ik}, x_{js}) = \frac{(1 - \sigma_0^2)}{nm}\left[\Lambda(\lambda) - \frac{(\lambda + a + a\sigma_0^2)}{(\lambda + a)(\lambda + a\sigma_0^2)}\right].$$

Therefore,

$$\mathsf{k}(\lambda, \omega) = m\sum_{k=0}^{n-1} \frac{1}{\lambda + a\sigma_0^2 + 4c\sin^2\left(\frac{k\pi}{n}\right)} - \frac{a(1 - \sigma_0^2)}{(\lambda + a)(\lambda + a\sigma_0^2)}.$$

Moreover, for any $i = 0, \ldots, m$, then

$$r_{\lambda,\omega}(x_0) = \frac{(1 - \sigma_0^2)}{\sigma_0^2(\lambda + a)}$$

and

$$r_{\lambda,\omega}(x_{ik}) = \frac{m}{1-\sigma_0^2}\sum_{k=0}^{n-1}\frac{1}{\lambda + a\sigma_0^2 + 4c\sin^2\left(\frac{k\pi}{n}\right)} - \frac{(\lambda + a + a\sigma_0^2)}{(\lambda + a)(\lambda + a\sigma_0^2)}.$$

In addition, for any $k, s = 1, \dots, n$, $i, j = 1, \dots, m$, $i \neq j$,

$$R_{\lambda,\omega}(x_{ik}, x_{is}) = \frac{nm\Big(U_{n-1}(p) - U_{n-1-|k-s|}(p) - U_{|k-s|-1}(p)\Big)}{c(1-\sigma_0^2)(T_n(p) - 1)}$$

$$R_{\lambda,\omega}(x_0, x_{js}) = \frac{m}{1-\sigma_0^2}\sum_{k=0}^{n-1}\frac{1}{\lambda + a\sigma_0^2 + 4c\sin^2\left(\frac{k\pi}{n}\right)} + \frac{\lambda}{\sigma_0^2(\lambda + a)(\lambda + a\sigma_0^2)}$$

$$R_{\lambda,\omega}(x_{ik}, x_{js}) = \frac{2m}{1-\sigma_0^2}\sum_{k=0}^{n-1}\frac{1}{\lambda + a\sigma_0^2 + 4c\sin^2\left(\frac{k\pi}{n}\right)}.$$

We must note that when $m = 1$, the m-star cycle is nothing but the wagon-wheel, and the above results coincide with those obtained in [6]; see also [1,4,8] for different approaches in the case $\lambda = 0$.

Proposition 3.5. *If* $p = 1 + \frac{\lambda + a\sigma_0^2}{2c}$, *then Green's function of a m-star path is given by*

$$G_q(x_0, x_0) = \frac{1-\sigma_0^2}{\lambda + a} + \Lambda(\lambda)\sigma_0^2$$

and for any $k, s = 1, \dots, n,\ i, j = 1, \dots, m,\ i \neq j$,

$$G_q(x_{ik}, x_{is}) = \frac{2c\Big(T_{n-|k-s|}(p) + T_{n-s-k+1}(p)\Big)}{(\lambda + a\sigma_0^2)(4c + \lambda + a\sigma_0^2)U_{n-1}(p)} + \frac{(1-\sigma_0^2)}{nm}\left[\Lambda(\lambda) - \frac{(\lambda + a + a\sigma_0^2)}{(\lambda + a)(\lambda + a\sigma_0^2)}\right],$$

$$G_q(x_0, x_{ik}) = \frac{\sigma_0\sqrt{1-\sigma_0^2}}{\sqrt{mn}}\left[\Lambda(\lambda) - \frac{1}{\lambda + a}\right],$$

$$G_q(x_{ik}, x_{js}) = \frac{(1-\sigma_0^2)}{nm}\left[\Lambda(\lambda) - \frac{(\lambda + a + a\sigma_0^2)}{(\lambda + a)(\lambda + a\sigma_0^2)}\right].$$

Therefore,

$$\mathsf{k}(\lambda, \omega) = m\sum_{k=1}^{n-1}\frac{1}{\lambda + a\sigma_0^2 + 4c\sin^2\left(\frac{k\pi}{2n}\right)} - \frac{a(1-\sigma_0^2)}{(\lambda + a)(\lambda + a\sigma_0^2)}.$$

Moreover, for any $i = 0, \dots, m$, *then*

$$r_{\lambda,\omega}(x_0) = \frac{(1-\sigma_0^2)}{\sigma_0^2(\lambda + a)}$$

and

$$r_{\lambda,\omega}(x_{ik}) = \frac{2cnm\Big(T_n(p) + T_{n+1-2k}(p)\Big)}{(1-\sigma_0^2)(\lambda + a\sigma_0^2)(4c + \lambda + a\sigma_0^2)U_{n-1}(p)} - \frac{(\lambda + a + a\sigma_0^2)}{(\lambda + a)(\lambda + a\sigma_0^2)}.$$

In addition, for any $k, s = 1, \dots, n,\ i, j = 1, \dots, m,\ i \neq j$,

$$R_{\lambda,\omega}(x_{ik}, x_{is}) = \frac{4cnm\Big(T_n(p) + T_{n+1-k-s}(p)(T_{|k-s|}(p) - 1) - T_{n-|k-s|}(p)\Big)}{(1-\sigma_0^2)(\lambda + a\sigma_0^2)(4c + \lambda + a\sigma_0^2)U_{n-1}(p)}$$

$$R_{\lambda,\omega}(x_0, x_{js}) = \frac{2cnm\Big(T_n(p) + T_{n+1-2s}(p)\Big)}{(1-\sigma_0^2)(\lambda + a\sigma_0^2)(4c + \lambda + a\sigma_0^2)U_{n-1}(p)} + \frac{\lambda}{\sigma_0^2(\lambda + a)(\lambda + a\sigma_0^2)}$$

$$R_{\lambda,\omega}(x_{ik}, x_{js}) = \frac{4cmn\Big(T_n(p) + T_{n+1-k-s}(p)T_{|k-s|}(p)\Big)}{(1-\sigma_0^2)(\lambda + a\sigma_0^2)(4c + \lambda + a\sigma_0^2)U_{n-1}(p)}.$$

We must note that when $m = 1$, the m-star path is nothing but the so-called *fan graph* and when $\lambda = 0$, the above results were obtained using a different approach in [1].

4. Cone networks

In this section we consider the *m-cone network* obtained by joining m singletons, say $\{x_1, \dots, x_m\}$, to a network Γ_0 with constant join conductance a. Consider $\omega \in \Omega(V)$ such that $\omega(x_i) = \sigma_1,\ 1 \le i \le m$, where $0 < \sigma_1 < \frac{1}{\sqrt{m}}$. Then, $\sigma_0 = \sqrt{1 - m\sigma_1^2}$, $\gamma_0 = \lambda + am\sigma_1^2$ and $\gamma_i = \gamma = \lambda + a(1 - m\sigma_1^2)$.

Proposition 4.1. *Green's function of a m-cone network is given by*

$$G_q(x, y) = G^0_{p_0}(x, y) + \left[\Lambda(\lambda) - \frac{(\lambda + a + ma\sigma_1^2)}{(\lambda + a)(\lambda + am\sigma_1^2)}\right]\omega(x)\omega(y)$$

when $x, y \in V_0$ and for any $i, j = 1, \dots, m$, $i \neq j$,

$$G_q(x_i, x_i) = \Lambda(\lambda) + \left[\Lambda(\lambda) - \frac{(\lambda + 2a - am\sigma_1^2)}{(\lambda + a)(\lambda + a - am\sigma_1^2)}\right]\sigma_1^2,$$

$$G_q(x, x_i) = \left[\Lambda(\lambda) - \frac{1}{\lambda + a}\right]\sigma_1\omega(x),$$

$$G_q(x_i, x_j) = \left[\Lambda(\lambda) - \frac{(\lambda + 2a - am\sigma_1^2)}{(\lambda + a)(\lambda + a - am\sigma_1^2)}\right]\sigma_1^2.$$

Therefore,

$$\mathsf{k}(\lambda, \omega) = \mathsf{k}_0(\gamma_0, \omega_0) + \frac{m - 1}{\lambda + a - am\sigma_1^2} + \frac{1}{\lambda + a}.$$

Moreover, for any $x \in V_0$ and any $i = 1, \dots, m$,

$$r_{\lambda,\omega}(x) = \frac{1}{1 - m\sigma_1^2}\left[r_{\gamma_0,\omega_0}(x) + \frac{m\sigma_1^2}{\lambda + a}\right] \quad \textit{and} \quad r_{\lambda,\omega}(x_i) = \frac{\lambda + a - \sigma_1^2(\lambda + 2a - am\sigma_1^2)}{\sigma_1^2(\lambda + a - am\sigma_1^2)(\lambda + a)}.$$

In addition, for any $x, y \in V_0$ and $i, j = 1, \dots, m$, $i \neq j$,

$$R_{\lambda,\omega}(x, y) = \frac{1}{1 - m\sigma_1^2} R_{\gamma_0,\omega_0}(x, y)$$

$$R_{\lambda,\omega}(x, x_i) = \frac{1}{1 - m\sigma_1^2}\left[r_{\gamma_0,\omega_0}(x) + \frac{(\lambda + a)(1 - m\sigma_1^2) + \lambda\sigma_1^2}{\sigma_1^2(\lambda + a - am\sigma_1^2)(\lambda + a)}\right]$$

$$R_{\lambda,\omega}(x_i, x_j) = \frac{2}{\sigma_1^2(\lambda + a - am\sigma_1^2)}.$$

To end this section let us consider two particular cases of m-cone networks; namely, the standard *m-cone* and the *m-fan* that are obtained when Γ_0 is a cycle or a path, respectively. We use the notation of the preceding section. Moreover, we consider that $\omega(x) = \sqrt{\frac{1 - m\sigma_1^2}{n}}$, for any $x \in V_0 = \{x_{01}, \dots, x_{0n}\}$.

Corollary 4.2. *If $p = 1 + \frac{\lambda + am\sigma_1^2}{2c}$, then Green's function of the standard m-cone is given by*

$$G_q(x_{0k}, x_{0s}) = \frac{U_{n-1-|k-s|}(p) + U_{|k-s|-1}(p)}{2c(T_n(p) - 1)} + \left[\Lambda(\lambda) - \frac{(\lambda + a + ma\sigma_1^2)}{(\lambda + a)(\lambda + am\sigma_1^2)}\right]\frac{(1 - m\sigma_1^2)}{n}$$

when $k, s = 1, \dots, n$ and for any $i, j = 1, \dots, m$, $i \neq j$,

$$G_q(x_i, x_i) = \Lambda(\lambda) + \left[\Lambda(\lambda) - \frac{(\lambda + 2a - am\sigma_1^2)}{(\lambda + a)(\lambda + a - am\sigma_1^2)}\right]\sigma_1^2,$$

$$G_q(x_{0k}, x_i) = \left[\Lambda(\lambda) - \frac{1}{\lambda + a}\right]\sigma_1\sqrt{\frac{1 - m\sigma_1^2}{n}},$$

$$G_q(x_i, x_j) = \left[\Lambda(\lambda) - \frac{(\lambda + 2a - am\sigma_1^2)}{(\lambda + a)(\lambda + a - am\sigma_1^2)}\right]\sigma_1^2.$$

Therefore,

$$\mathsf{k}(\lambda,\omega)=\sum_{k=1}^{n-1}\frac{1}{\lambda+am\sigma_1^2+4c\sin^2\left(\frac{k\pi}{n}\right)}+\frac{m-1}{\lambda+a-am\sigma_1^2}+\frac{1}{\lambda+a}.$$

Moreover, for any $k=1,\dots,n$ and any $i=1,\dots,m$,

$$r_{\lambda,\omega}(x_{0k})=\frac{1}{1-m\sigma_1^2}\left[\sum_{k=1}^{n-1}\frac{1}{\lambda+am\sigma_1^2+4c\sin^2\left(\frac{k\pi}{n}\right)}+\frac{m\sigma_1^2}{\lambda+a}\right]$$

and

$$r_{\lambda,\omega}(x_i)=\frac{\lambda+a-\sigma_1^2(\lambda+2a-am\sigma_1^2)}{\sigma_1^2(\lambda+a-am\sigma_1^2)(\lambda+a)}.$$

In addition, for any $k,s=1,\dots,n$ and $i,j=1,\dots,m$, $i\neq j$,

$$R_{\lambda,\omega}(x_{0k},x_{0s})=\frac{n\Big(U_{n-1}(p)-U_{n-1-|k-s|}(p)-U_{|k-s|-1}(p)\Big)}{c(1-m\sigma_1^2)(T_n(p)-1)}$$

$$R_{\lambda,\omega}(x_{0k},x_i)=\frac{1}{1-m\sigma_1^2}\left[\sum_{k=1}^{n-1}\frac{1}{\lambda+am\sigma_1^2+4c\sin^2\left(\frac{k\pi}{n}\right)}+\frac{(\lambda+a)(1-m\sigma_1^2)+\lambda\sigma_1^2}{\sigma_1^2(\lambda+a-am\sigma_1^2)(\lambda+a)}\right]$$

$$R_{\lambda,\omega}(x_i,x_j)=\frac{2}{\sigma_1^2(\lambda+a-am\sigma_1^2)}.$$

Corollary 4.3. *If $p=1+\frac{\lambda+am\sigma_1^2}{2c}$, then Green's function of the m-fan is given by*

$$G_q(x_{0k},x_{0s})=\frac{T_{n-|k-s|}(p)+T_{n-s-k+1}(p)}{(\lambda+am\sigma_1^2)(p+1)U_{n-1}(p)}+\left[\Lambda(\lambda)-\frac{(\lambda+a+ma\sigma_1^2)}{(\lambda+a)(\lambda+am\sigma_1^2)}\right]\frac{(1-m\sigma_1^2)}{n}$$

when $k,s=1,\dots,n$ and for any $i,j=1,\dots,m$, $i\neq j$,

$$G_q(x_i,x_i)=\Lambda(\lambda)+\left[\Lambda(\lambda)-\frac{(\lambda+2a-am\sigma_1^2)}{(\lambda+a)(\lambda+a-am\sigma_1^2)}\right]\sigma_1^2,$$

$$G_q(x_{0k},x_i)=\left[\Lambda(\lambda)-\frac{1}{\lambda+a}\right]\sigma_1\sqrt{\frac{1-m\sigma_1^2}{n}},$$

$$G_q(x_i,x_j)=\left[\Lambda(\lambda)-\frac{(\lambda+2a-am\sigma_1^2)}{(\lambda+a)(\lambda+a-am\sigma_1^2)}\right]\sigma_1^2.$$

Therefore,

$$\mathsf{k}(\lambda,\omega)=\sum_{k=1}^{n-1}\frac{1}{\lambda+am\sigma_1^2+4c\sin^2\left(\frac{k\pi}{2n}\right)}+\frac{m-1}{\lambda+a-am\sigma_1^2}+\frac{1}{\lambda+a}.$$

Moreover, for any $k=1,\dots,n$ and any $i=1,\dots,m$,

$$r_{\lambda,\omega}(x_{0k})=\frac{1}{\lambda+am\sigma_1^2}\left[\frac{n\Big(T_n(p)+T_{n+1-2k}(p)\Big)}{(1-m\sigma_1^2)(p+1)U_{n-1}(p)}-\frac{(\lambda+a+am\sigma_1^2)}{\lambda+a}\right]$$

and

$$r_{\lambda,\omega}(x_i)=\frac{\lambda+a-\sigma_1^2(\lambda+2a-am\sigma_1^2)}{\sigma_1^2(\lambda+a-am\sigma_1^2)(\lambda+a)}.$$

In addition, for any $k,s=1,\dots,n$ and $i,j=1,\dots,m$, $i\neq j$,

$$R_{\lambda,\omega}(x_{0k},x_{0s})=\frac{2n\Big(T_n(p)+T_{n+1-k-s}(p)[T_{|k-s|}(p)-1]-T_{n-|k-s|}(p)\Big)}{(\lambda+am\sigma_1^2)(1-m\sigma_1^2)(p+1)U_{n-1}(p)}$$

$$R_{\lambda,\omega}(x_{0k}, x_i) = \frac{n\Big(T_n(p) + T_{n+1-2k}(p)\Big)}{(1 - m\sigma_1^2)(\lambda + am\sigma_1^2)(p+1)U_{n-1}(p)} + \frac{1}{\sigma_1^2(\lambda + a - am\sigma_1^2)} - \frac{a(2\lambda + a)}{(\lambda + am\sigma_1^2)(\lambda + a - am\sigma_1^2)(\lambda + a)}$$

$$R_{\lambda,\omega}(x_i, x_j) = \frac{2}{\sigma_1^2(\lambda + a - am\sigma_1^2)}.$$

When $\lambda = 0$, $a = n + m$, $c = 1$ and $\sigma_1 = \sqrt{\frac{1}{n+m}}$, the value of the Kirchhoff index obtained in Corollaries 4.2 and 4.3, coincide with those obtained in [12] Formula (3.16) and (3.17), respectively.

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